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Closed embeddings of Hilbert spaces

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ABSTRACT

Motivated by questions related to embeddings of homogeneous Sobolev spaces and to comparison of function spaces and operator ranges, we introduce the notion of closely embedded Hilbert spaces as an extension of that of continuous embedding of Hilbert spaces. We show that this notion is a special case of that of Hilbert spaces induced by unbounded positive selfadjoint operators that corresponds to kernel operators in the sense of L. Schwartz. Certain canonical representations and characterizations of uniqueness of closed embeddings are obtained. We exemplify these constructions by closed, but not continuous, embeddings of Hilbert spaces of holomorphic functions. An application to the closed embedding of a homogeneous Sobolev space on $\mathbb{R}^n$ in $L_2(\mathbb{R}^n)$, based on the singular integral operator associated to the Riesz potential, and a comparison to the case of the singular integral operator associated to the Bessel potential are also presented. As a second application we show that a closed embedding of two operator ranges corresponds to absolute continuity, in the sense of T. Ando, of the corresponding kernel operators.

1. Introduction

Operator ranges (para-closed subspaces, in the terminology of C. Foiaş [16]) of Hilbert spaces have been considered for a long time, as a more flexible substitute of closed subspaces of Hilbert spaces, by J. Dixmier [12,13], G.W. Mackey [23], L. de Branges and J. Rovnyak [8] and many others, as shown in the semi-expository paper of P.A. Fillmore and J.P. Williams [15]. A linear manifold $\mathcal{L}$ of a Hilbert space $\mathcal{H}$ is an operator range if there exists T a bounded linear operator $T:\mathcal{G}\to\mathcal{H}$ such that $\mathcal{L}=\mathrm{Ran}(T)$. Replacing T by a closed operator does not change the definition and the same is true if instead we require $\mathcal{L}=\mathrm{Dom}(T)$ for some closed operator. What is even more interesting is that this notion coincides with that of a Hilbert space continuously embedded in $\mathcal{H}$, that is, a Hilbert space $\mathcal{H}_+\subseteq\mathcal{H}$, as vector spaces, such that the embedding $j_+:\mathcal{H}_+\hookrightarrow\mathcal{H}$ is a bounded linear operator. Letting $\mathrm{Hilb}(\mathcal{H})$ denote the collection of all Hilbert spaces closely embedded in a given Hilbert space $\mathcal{H}$, it can be proven that $\mathrm{Hilb}(\mathcal{H})$ is in a bijective correspondence with $\mathcal{B}(\mathcal{H})^+$, the convex cone of all nonnegative bounded operators in $\mathcal{H}$. This correspondence is given explicitly by associating the operator $A=j_+j_+^*$ to any Hilbert space $\mathcal{H}_+$ that is continuously embedded in $\mathcal{H}$, where $j_+:\mathcal{H}_+\hookrightarrow\mathcal{H}$, and it was obtained by L. Schwartz in [28] even in the more general case when $\mathcal{H}$ is replaced by a quasi-complete, Hausdorff separated, locally convex space, with suitably changed definitions of positivity. Actually, L. Schwartz showed that his theory is equivalent with that of reproducing kernel Hilbert spaces, which has a longer and more substantial history, e.g. see N. Aronszajn [5] and the bibliography cited there. In this respect, the operator $A=j_+j_+^*$ is called the kernel operator of $\mathcal{H}_+$ and this provides a very

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powerful tool for the study of $\mathrm{Hilb}(\mathcal{H})$: for example, if $\mathcal{L}$ and $\mathcal{R}$ are operator ranges in $\mathcal{H}$ then both $\mathcal{L} \cap \mathcal{R}$ and $\mathcal{L} + \mathcal{R}$ are operator ranges. Also, it is known, e.g. see [5], that reproducing kernel Hilbert spaces have an interesting automatic continuity property: once $\mathcal{L} \subseteq \mathcal{R}$, for two reproducing Hilbert spaces $\mathcal{L}$ and $\mathcal{R}$ of functions on the same base set X, then the embedding is topological. This feature is shared by operator ranges as well and it can be put into a more concrete perspective using the connections between range inclusions, majorizations, and factorizations of bounded operators, as illustrated by a simple but very useful theorem that was first made explicit by R.G. Douglas [14].

Keeping the discussion within the framework of Hilbert spaces, continuous embedding is an important topic in the theory of Sobolev spaces, e.g. considering the continuous embedding $W_2^l(\Omega) \subseteq L_2(\Omega)$, that can be proved for certain domains Ω in $\mathbb{R}^n$, see [1,24] and the bibliography cited there. In addition, many results within the theory of spaces of holomorphic functions like Hardy, Bergman, Bloch, Besov, and their generalizations, are referring to a variety of continuous embeddings. However, to our knowledge, only bounded kernel operators have so far been considered, but, from the perspective of applications to linear operators of interest in mathematical physics it is necessary to consider unbounded kernel operators. To substantiate this assertion, let us mention that the homogeneous versions of Sobolev spaces cannot be continuously embedded in the ambient L_2 spaces and that similar phenomena appear in different theories of Hilbert spaces of holomorphic functions as well. In addition, investigations on a noncommutative Radon–Nikodym derivative have been related to the notion of absolute continuity for bounded nonnegative operators, cf. T. Ando [4] and proved to be of interest for the theory of quantum operations, e.g. see [17] and the bibliography cited there, make natural to ask whether there is any correspondent of this absolute continuity in terms of the associated operator ranges. All these questions make the motivations for this research and, to some of them, we can provide reasonable answers in this article.

In this paper we extend the notion of continuous embedding of Hilbert spaces to that of closed embedding in such a way that unbounded kernel operators are allowed, and we investigate its connection with operator ranges, its properties and especially uniqueness properties. We exemplify these constructions by pointing out closed embeddings that appear when comparing some Hilbert spaces of holomorphic functions on the disk (Hardy, Dirichlet, Bergman). As main applications we prove the closed embedding for certain homogeneous Sobolev spaces on the semi-axis and the Euclidean space $\mathbb{R}^n$ into the ambient L_2 space and we obtain an equivalent characterization of absolutely continuous bounded kernel operators in terms of closed embeddings. In this enterprise, our approach starts from previous investigations on the abstract notion of Hilbert spaces induced by unbounded operators, as considered in our article [9], and we show that closed embedding is a special representation of a Hilbert space induced by a positive selfadjoint operator within the ambient Hilbert space. This connection allows us to obtain a variant of the lifting theorem for closely embedded Hilbert spaces as well.

Other questions of this kind as closed embeddings of Krein spaces and their application to energy space representations associated to free particle Dirac operators, closed embeddings of Hilbert spaces of holomorphic functions of several complex variable, as well as for Hilbert spaces of harmonic functions, will be considered in subsequent articles.

2. Preliminaries

In order to extend the notion of continuously embedded Hilbert spaces to a case when the kernel operators can be unbounded, we first analyze the notion of operator range from this point of view by providing a universal model space, from where we will derive the definition of closely embedded Hilbert spaces in Section 3. On the other hand, since the notion of closely embedded Hilbert space will be a particular representation of that of induced Hilbert space, we first recall the basic notions and facts about this.

In the following we rely on the spectral properties of unbounded selfadjoint operators and the polar decompositions for closed densely defined operators in Hilbert space, e.g. see [7,19,27]. A few words about notation: if $\mathcal{G}$ and $\mathcal{H}$ are Hilbert spaces, then $\mathcal{B}(\mathcal{G},\mathcal{H})$ denotes the set of bounded operators $T:\mathcal{G} \to \mathcal{H}$, while $\mathcal{C}(\mathcal{G},\mathcal{H})$ denotes the set of all closed and densely defined operators from $\mathcal{G}$ and valued in $\mathcal{H}$.

2.1. Hilbert spaces induced by nonnegative selfadjoint operators

In this section we recall the definitions and the basic results on induced Hilbert spaces, cf. [9]. Let $\mathcal{H}$ be a Hilbert space and A a densely defined nonnegative operator in $\mathcal{H}$ (in this paper, the nonnegativity of an operator A means $\langle Ax, x\rangle_{\mathcal{H}} \geqslant 0$ for all $x \in \mathrm{Dom}(A)$). A pair $(\mathcal{K}, \Pi)$ is called a *Hilbert space induced* by A if:

(ihs1) $\mathcal{K}$ is a Hilbert space;
(ihs2) Π is a linear operator with domain $\mathrm{Dom}(\Pi) \supseteq \mathrm{Dom}(A)$ and range in $\mathcal{K}$;
(ihs3) $\Pi\,\mathrm{Dom}(A)$ is dense in $\mathcal{K}$;
(ihs4) $\langle \Pi x, \Pi y\rangle_{\mathcal{K}} = \langle Ax, y\rangle_{\mathcal{H}}$ for all $x \in \mathrm{Dom}(A)$ and all $y \in \mathrm{Dom}(\Pi)$.

Given any densely defined nonnegative operator A, Hilbert spaces induced by A always exist by an obvious quotient-completion procedure. In addition, they are essentially unique in the following sense: two Hilbert spaces $(\mathcal{K}_i, \Pi_i)$, $i = 1, 2$, induced by the same operator A, are called *unitary equivalent* if there exists a unitary operator $U \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that $U\Pi_1 = \Pi_2$.

Remark 2.1. In the case of a nonnegative selfadjoint operator, the quotient-completion construction can be made more explicit. Thus, if A is a nonnegative selfadjoint operator in the Hilbert space $\mathcal{H}$, then $A^{1/2}$ exists as a nonnegative selfadjoint operator in $\mathcal{H}$, $\mathrm{Dom}(A^{1/2}) \supseteq \mathrm{Dom}(A)$ and $\mathrm{Dom}(A)$ is a core of $A^{1/2}$. In particular we have

$$\langle Ax, y\rangle_{\mathcal{H}} = \big\langle A^{1/2}x, A^{1/2}y\big\rangle_{\mathcal{H}}, \quad x \in \mathrm{Dom}(A),\ y \in \mathrm{Dom}\big(A^{1/2}\big),$$

which shows that we can consider the seminorm $\|A^{1/2}\cdot\|$ on $\mathrm{Dom}(A)$ and make the quotient completion with respect to this seminorm in order to get a Hilbert space $\mathcal{K}_A$. We denote by Π_A the corresponding canonical operator, more precisely, Π_A is the composition of the canonical map $\mathcal{D} \mapsto \mathcal{D}/\mathrm{Ker}(A)$ with the embedding into $\mathcal{K}_A$. Then it is easy to see that $(\mathcal{K}_A, \Pi_A)$ is a Hilbert space induced by A.

The main result of [9] is the following lifting theorem that generalizes lifting theorems as in [21,26,22,11]:

Theorem 2.2. *Let A and B be nonnegative selfadjoint operators in the Hilbert spaces $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, and let $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{K}_B, \Pi_B)$ be the Hilbert spaces induced by A and respectively B. For any operators $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle Bx, Ty\rangle_{\mathcal{H}_2} = \langle Sx, Ay\rangle_{\mathcal{H}_1}, \quad x \in \mathrm{Dom}(B),\ y \in \mathrm{Dom}(A), \tag{2.1}$$

there exist uniquely determined operators $\widetilde{T} \in \mathcal{B}(\mathcal{K}_A, \mathcal{K}_B)$ and $\widetilde{S} \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_A)$ such that $\widetilde{T}\Pi_A x = \Pi_B Tx$ for all $x \in \mathrm{Dom}(A)$, $\widetilde{S}\Pi_B y = \Pi_A Sy$ for all $y \in \mathrm{Dom}(B)$, and

$$\langle \widetilde{S}h, k\rangle_{\mathcal{K}_A} = \langle h, \widetilde{T}k\rangle_{\mathcal{K}_B}, \quad h \in \mathcal{K}_B,\ k \in \mathcal{K}_A. \tag{2.2}$$

2.2. The Hilbert space $\mathcal{R}(T)$

We first present a construction of Hilbert spaces associated to ranges of general linear operators that will provide the model for the closed embedded Hilbert space.

Let T be a linear operator acting from a Hilbert space $\mathcal{G}$ to another Hilbert space $\mathcal{H}$ and such that its kernel $\mathrm{Ker}(T)$ is closed. Introduce a pre-Hilbert space structure on $\mathrm{Ran}(T)$ by the positive definite inner product $\langle\cdot,\cdot\rangle_T$ defined by

$$\langle u, v\rangle_T = \langle x, y\rangle_{\mathcal{G}} \tag{2.3}$$

for all $u = Tx$, $v = Ty$, $x, y \in \mathrm{Dom}(T)$ such that $x, y \perp \mathrm{Ker}(T)$. Let $\mathcal{R}(T)$ be the completion of the pre-Hilbert space $\mathrm{Ran}(T)$ with respect to the corresponding norm $\|\cdot\|_T$, where $\|u\|_T^2 := \langle u, u\rangle_T$, for $u \in \mathrm{Ran}(T)$. The inner product and the norm on $\mathcal{R}(T)$ will be denoted by $\langle\cdot,\cdot\rangle_T$ and $\|\cdot\|_T$ throughout.

We first justify the assumption on the kernel of T.

Example 2.3. Let $\mathcal{H} = L_2[0,1]$ and consider the operator T in $\mathcal{H}$ with $\mathrm{Dom}(T) = C[0,1]$ defined by $(Tx) = x(0)1$, where 1 is the function identically equal to 1 on $[0,1]$. Because $\mathrm{Ker}(T) = \{x \in C[0,1] \mid x(0) = 0\}$ there exists no nontrivial vector in $\mathrm{Dom}(T) \ominus \mathrm{Ker}(T)$, and since $\mathrm{Ker}(T)$ is dense in $L_2[0,1]$ it follows $T(\mathrm{Dom}(T) \ominus \mathrm{Ker}(T)) = \{0\}$. On the other hand, $\mathrm{Ran}(T) = \mathbb{C}1$ is nontrivial. In addition, let us remark that T is not closable: the sequence $x_n(t) = (1-t)^n$ has the properties $x_n \in \mathrm{Dom}(T)$ and $x_n \to 0$ but $Tx_n = 1$ for all $n \in \mathbb{N}$.

Now we set a topological property.

Remark 2.4. With the notation as before, $u \in \mathcal{R}(T)$ if and only if there exists a sequence $x_n \in \mathrm{Dom}(T)$, $x_n \perp \mathrm{Ker}(T)$ such that $\|Tx_n - u\|_T \to 0$. In addition, in this case there exists $x \in \mathcal{G}$ such that $\|x_n - x\|_{\mathcal{G}} \to 0$ as $n \to \infty$.

Indeed, let $u \in \mathcal{R}(T)$. Since the set $\mathrm{Ran}(T)$ is dense in $\mathcal{R}(T)$ we can find a sequence (u_n) of vectors in $\mathrm{Ran}(T)$ that converges to u in $\mathcal{R}(T)$. To each element u_n there corresponds $x_n \in \mathrm{Dom}(T)$ such that $x_n \perp \mathrm{Ker}(T)$ and $u_n = Tx_n$ for all $n = 1, 2, \ldots$. Due to the fact that

$$\|x_n - x_m\|_{\mathcal{G}} = \|u_n - u_m\|_T \to 0 \quad \text{as } n, m \to \infty,$$

we have that (x_n) is a Cauchy sequence in $\mathcal{G}$. Thus, (x_n) is convergent to some element $x \in \mathcal{G}$, and the assertion follows.

The converse implication is clear.

Further, consider the *embedding operator* $j_T : \mathrm{Dom}(j_T)(\subseteq \mathcal{R}(T)) \to \mathcal{H}$ with domain $\mathrm{Dom}(j_T) = \mathrm{Ran}(T)$ defined by

$$j_T u = u, \quad u \in \mathrm{Dom}(j_T) = \mathrm{Ran}(T). \tag{2.4}$$

Another way of viewing the definition of the Hilbert space $\mathcal{R}(T)$ is by means of a certain factorization of T.

Lemma 2.5. *Let T be a linear operator with domain dense in the Hilbert space $\mathcal{G}$, valued in the Hilbert space $\mathcal{H}$, and with closed kernel. We consider the Hilbert space $\mathcal{R}(T)$ and the embedding j_T defined as in* (2.3) *and, respectively,* (2.4)*. Then, there exists a unique coisometry $U_T \in \mathcal{B}(\mathcal{G}, \mathcal{R}(T))$, such that* $\mathrm{Ker}(U_T) = \mathrm{Ker}(T)$ *and $T = j_T U_T$.*

Proof. Let $U_T : \mathrm{Dom}(T)(\subseteq \mathcal{G}) \to \mathcal{R}(T)$ be defined by $U_T x = Tx$ for arbitrary $x \in \mathrm{Dom}(T)$. Then $U_T x = 0$ for $x \in \mathrm{Ker}(T)$ and, if $x \perp \mathrm{Ker}(T)$ then $\|U_T x\|_T = \|Tx\|_T = \|x\|_{\mathcal{G}}$. Since $\mathrm{Ran}(U_T) = \mathrm{Ran}(T)$ is, by definition, dense in $\mathcal{R}(T)$, and $\mathrm{Dom}(T)$ is, by assumption, dense in $\mathcal{G}$, it follows that U_T can be extended uniquely to a coisometry $U_T \in \mathcal{B}(\mathcal{G}, \mathcal{R}(T))$ such that $\mathrm{Ker}(U_T) = \mathrm{Ker}(T)$ and for all $x \in \mathrm{Dom}(T)$ we have $Tx = U_T x = j_T U_T x$, hence $T \subseteq j_T U_T$.

In addition, since $\mathrm{Ker}(T)$ is closed it follows that for arbitrary $x \in \mathrm{Dom}(j_T U_T)$ we have $U_T x \in \mathrm{Dom}(j_T) = \mathrm{Ran}(T)$, hence there exists $y \in \mathrm{Dom}(T)$ such that $U_T x = Ty = U_T y$. Therefore, $x - y \in \mathrm{Ker}(U_T) = \overline{\mathrm{Ker}(T)} = \mathrm{Ker}(T) \subseteq \mathrm{Dom}(T)$ and hence $x \in \mathrm{Dom}(T)$. This shows that the converse inclusion $j_T U_T \subseteq T$ holds, too.

The uniqueness of the coisometry U_T is clear, from its properties. □

Remark 2.6. The assumption in Lemma 2.5 that T is densely defined is not so important; if this is not the case then U_T must have a larger kernel only, in order to keep it unique. More precisely, $\mathrm{Ker}(U_T) = \mathrm{Ker}(T) \oplus (\mathcal{G} \ominus \mathrm{Dom}(T))$ and, consequently, $TP_{\overline{\mathrm{Dom}(T)}} \subseteq j_T U_T$, which turns out to be an equality since $\mathrm{Ker}(T)$ is supposed to be a closed subspace in $\mathcal{G}$.

The most interesting situation, from our point of view, is when the embedding operator has some closability properties. The proof of the next proposition is an easy consequence of Lemma 2.5.

Proposition 2.7. *Let T be an operator densely defined in $\mathcal{G}$, with range in $\mathcal{H}$, and with closed kernel. With the notation as before, the operator T is closed if and only if the embedding operator j_T is closed.*

In the following we consider the relation between taking the closures of T and that of the associated embedding j_T. Given a closable operator T we denote by $\overline{T}$ its closure.

Proposition 2.8. *Let T be an operator from $\mathcal{G}$ to $\mathcal{H}$ with closed kernel.*

(1) *The operator T is closable if and only if j_T is closable.*
(2) *Assume that T is closable. Then*:
 (i) *With the notation as in Lemma* 2.5 *we have $\overline{T} \subseteq \bar{j}_T U_T$ and $\bar{j}_T = \overline{T} U_T^*$.*
 (ii) *The closure extension $\bar{j}_T$ of the embedding operator j_T is an injective operator if and only if* $\mathrm{Ker}(\overline{T}) = \mathrm{Ker}(T)$.
 (iii) *Assume that* $\mathrm{Ker}(\overline{T}) = \mathrm{Ker}(T)$. *Then the identity mapping* $\mathrm{Ran}(T)(\subseteq \mathcal{R}(T)) \to \mathrm{Ran}(\overline{T})(\subseteq \mathcal{R}(\overline{T}))$ *extends uniquely to a unitary operator $V_T : \mathcal{R}(T) \to \mathcal{R}(\overline{T})$ and $\bar{j}_T = j_{\overline{T}} V_T$.*

Proof. (1) This is a consequence of Lemma 2.5.

(2) Assume that T is closable, hence j_T is the same.

(i) As a consequence of Lemma 2.5 we have $T = j_T U_T$ and, taking into account that U_T is bounded, it follows that $\overline{T} \subseteq \bar{j}_T U_T$.

On the other hand, since U_T is a coisometry with $\mathrm{Ker}(U_T) = \mathrm{Ker}(T)$ it follows that $j_T = T U_T^*$ whence, as before, it follows that $\bar{j}_T \subseteq \overline{T} U_T^*$. Since $U_T U_T^* = I$, from $\overline{T} \subseteq \bar{j}_T U_T$ it follows that $\overline{T} U_T^* \subseteq \bar{j}_T$, hence $\bar{j}_T = \overline{T} U_T^*$.

(ii) Assume that $\bar{j}_T$ is injective. Since $\bar{j}_T = \overline{T} U_T^*$ and U_T is a coisometry with $\mathrm{Ker}(U_T) = \mathrm{Ker}(T)$, it follows that $\mathrm{Ker}(\overline{T}) = \mathrm{Ker}(T)$.

Conversely, if $\mathrm{Ker}(\overline{T}) = \mathrm{Ker}(T)$, then we use again that $\bar{j}_T = \overline{T} U_T^*$ and that U_T is a coisometry with $\mathrm{Ker}(U_T) = \mathrm{Ker}(T)$ to conclude that $\bar{j}_T$ is injective.

(iii) Assume also that $\mathrm{Ker}(\overline{T}) = \mathrm{Ker}(T)$. Then $\overline{T} = \bar{j}_T U_T$. On the other hand, by Lemma 2.5 we have $\overline{T} = j_{\overline{T}} U_{\overline{T}}$. Letting $V_T = U_{\overline{T}} U_T^* \in \mathcal{B}(\mathcal{R}(T), \mathcal{R}(\overline{T}))$, it follows that V_T is unitary, that $V_T | \mathrm{Ran}(T)$ is the identity operator, and that $\bar{j}_T = j_{\overline{T}} V_T$. □

We now present a sequence of remarks in order to clarify the special situation when the completion $\mathcal{R}(T)$ can be realized within the space $\mathcal{H}$. The proofs are simple applications of either Lemma 2.5 or Proposition 2.8.

Remarks 2.9. Let T be a linear operator from $\mathcal{G}$ to $\mathcal{H}$, with closed kernel, and consider the construction as in (2.3).

(a) If $\overline{\mathrm{Dom}(T)} = \mathrm{Dom}(T)$ then $\mathcal{R}(T) = \mathrm{Ran}(T)$. Indeed for any $u \in \mathcal{R}(T)$ there exists a sequence (x_n), $x_n \in \mathrm{Dom}(T)$, $x_n \perp \mathrm{Ker}(T)$, such that $\|Tx_n - u\|_T \to 0$ as $n \to \infty$. An argument similar to that used above shows that (x_n) is convergent to an element $x \in \overline{\mathrm{Dom}(T)}$ $(= \mathrm{Dom}(T))$. Evidently, $x \perp \mathrm{Ker}(T)$ and, hence

$$\|Tx_n - Tx\|_T = \|x_n - x\|_{\mathcal{G}} \to 0.$$

Thus $u = Tx \in \mathrm{Ran}(T)$, and the assertion follows.

(b) T is a bounded operator on its domain if and only if the embedding operator j_T is bounded on $\mathrm{Ran}(T)$. Moreover, their bounds are the same.

(c) As in (b), let T be a bounded operator on its domain $\mathrm{Dom}(T)$. In this case $\mathcal{R}(\overline{T}) = \mathrm{Ran}(\overline{T})$ (cf. (a)), and the bounded extension $\bar{j}_T$ on $\mathcal{R}(T)$ maps the space $\mathcal{R}(T)$ onto $\mathrm{Ran}(\overline{T})$, i.e. $\mathrm{Ran}(\bar{j}_T) = \mathrm{Ran}(\overline{T})$. However, $\mathrm{Ker}(\bar{j}_T) \neq \{0\}$ if $\mathrm{Ker}(\overline{T}) \neq \mathrm{Ker}(T)$. In this case the space $\mathcal{R}(T)$ cannot be realized naturally by elements of $\mathrm{Ran}(T)$. In opposite case, that is when $\mathrm{Ker}(\overline{T}) = \mathrm{Ker}(T)$, $\mathrm{Ker}(\bar{j}_T) = \{0\}$ and $\mathcal{R}(T) = \mathrm{Ran}(\overline{T})$ $(= \mathcal{R}(\overline{T}))$.

2.3. Ranges of unbounded operators

In this subsection we consider some extensions of characterizations of ranges of bounded operators and related results. The next result generalizes a theorem of Yu.L. Shmulyan [29] and similar results of L. de Branges and J. Rovnyak [8], to the case of closed densely defined operators between Hilbert spaces.

Theorem 2.10. *Let $T \in \mathcal{C}(\mathcal{G},\mathcal{H})$ be nonzero and $u \in \mathcal{H}$. Then $u \in \operatorname{Ran}(T)$ if and only if there exists $\mu_u \geqslant 0$ such that $|\langle u, v\rangle_{\mathcal{H}}| \leqslant \mu_u \|T^*v\|_{\mathcal{G}}$ for all $v \in \operatorname{Dom}(T^*)$. Moreover, if $u \in \operatorname{Ran}(T)$ then*

$$\|u\|_T = \sup\left\{\frac{|\langle u, v\rangle_{\mathcal{H}}|}{\|T^*v\|_{\mathcal{G}}} \mid v \in \operatorname{Dom}(T^*),\ \|T^*v\|_{\mathcal{G}} \neq 0\right\},$$

where $\|\cdot\|_T$ is the norm associated to the inner product defined as in (2.3)*.*

Proof. Let $u \in \operatorname{Ran}(T)$. Then $u = Tx$ for some $x \in \operatorname{Dom}(T)$ and, for arbitrary $v \in \operatorname{Dom}(T^*)$, we have

$$\left|\langle u, v\rangle_{\mathcal{H}}\right| = \left|\langle Tx, v\rangle_{\mathcal{H}}\right| = \left|\langle x, T^*v\rangle_{\mathcal{G}}\right| \leqslant \|x\|_{\mathcal{G}}\left\|T^*v\right\|_{\mathcal{G}},$$

and take $\mu_u = \|x\|_{\mathcal{G}}$.

Conversely, if for some $u \in \mathcal{H}$ there exists $\mu_u \geqslant 0$ such that $|\langle u, v\rangle_{\mathcal{H}}| \leqslant \mu_u\|T^*v\|_{\mathcal{G}}$ for all $v \in \operatorname{Dom}(T^*)$, then the antilinear functional $\operatorname{Ran}(T^*) \ni T^*v \mapsto \varphi_u(T^*v) := \langle u, v\rangle_{\mathcal{H}}$ is well defined and bounded and hence by the Hahn–Banach theorem φ_u can be extended to a bounded antilinear functional on $\mathcal{G}$, also denoted by φ_u, with the same norm. Then, according to the F. Riesz representation theorem there exists a unique $y \in \mathcal{G}$ such that $\varphi_u(x) = \langle y, x\rangle_{\mathcal{G}}$ for all $x \in \mathcal{G}$. In particular, $\varphi_u(T^*v) = \langle y, T^*v\rangle_{\mathcal{G}}$, for all $v \in \operatorname{Dom}(T^*)$. Therefore, $y \in \operatorname{Dom}(T^{**}) = \operatorname{Dom}(T)$ and $u = T^{**}y = Ty \in \operatorname{Ran}(T)$.

Further, if $x \perp \operatorname{Ker}(T)$ then

$$\begin{aligned}\|u\|_T = \|x\|_{\mathcal{G}} &= \sup\left\{\frac{|\langle x, T^*v\rangle_{\mathcal{G}}|}{\|T^*v\|_{\mathcal{G}}} \mid v \in \operatorname{Dom}(T^*),\ \|T^*v\|_{\mathcal{G}} \neq 0\right\}\\ &= \sup\left\{\frac{|\langle u, v\rangle_{\mathcal{H}}|}{\|T^*v\|_{\mathcal{H}}} \mid v \in \operatorname{Dom}(T^*),\ \|T^*v\|_{\mathcal{G}} \neq 0\right\}. \qquad \square\end{aligned}$$

Range inclusion of operators is a classical issue related to majorization and factorization. The next result is essentially from R.G. Douglas [14], but there a different order relation was used. To be more precise, given two positive selfadjoint operators A and B on the same Hilbert space $\mathcal{H}$, the *form order relation* $A \leqslant B$ means that $\operatorname{Dom}(B^{1/2}) \subseteq \operatorname{Dom}(A^{1/2})$ and for all $x \in \operatorname{Dom}(B^{1/2})$ we have $\|A^{1/2}x\| \leqslant \|B^{1/2}x\|$. Since the differences in proof are minor we omit it.

Theorem 2.11. *Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces and let A and B be two closed densely defined operators from $\mathcal{G}$ into $\mathcal{H}$.*

(i) *If $AA^* \leqslant \mu^2BB^*$ for some $\mu \geqslant 0$ then $A \subseteq BC$ for some operator $C \in \mathcal{B}(\mathcal{G})$ with $\|C\| \leqslant \mu$. In addition, C can be chosen such that*:
   (1) $\|C\| = \inf\{\nu \geqslant 0 \mid AA^* \leqslant \nu^2BB^*\}$,
   (2) $\operatorname{Ker}(A) = \operatorname{Ker}(C)$,
   (3) $\operatorname{Ran}(C) \subseteq \overline{\operatorname{Ran}(B^*)}$,
   and it is uniquely determined with these three properties.
   On the other hand, if A is bounded and $AA^* \leqslant \mu^2BB^*$ for some $\mu \geqslant 0$, then $A = BC$ for some operator $C \in \mathcal{B}(\mathcal{G})$ with $\|C\| \leqslant \mu$.
(ii) *If $A \subseteq BC$ for some operator C then* $\operatorname{Ran}(A) \subseteq \operatorname{Ran}(B)$.
(iii) *If* $\operatorname{Ran}(A) \subseteq \operatorname{Ran}(B)$, *then $A = BC$ for some linear operator C densely defined in $\mathcal{G}$ such that $\|Cx\|^2 \leqslant M(\|x\|^2 + \|Ax\|^2)$ for all $x \in \operatorname{Dom}(C)$. If, in addition, A is bounded, then C can be chosen bounded, while if, in addition, B is bounded, then C can be chosen closed.*

Remark 2.12. For the moment we exemplify the use of Theorem 2.11 by yet another proof of Theorem 2.10. For $T \in \mathcal{C}(\mathcal{G},\mathcal{H})$ let $x \in \mathcal{H}$ be nontrivial. Then P, the orthogonal projection onto $\mathbb{C}x$ (the subspace of $\mathcal{H}$ spanned by x), is

$$Py = \frac{\langle y, x\rangle}{\|x\|^2}x, \quad y \in \mathcal{H}.$$

Thus, by Theorem 2.11, $x \in \operatorname{Ran}(T)$ if and only if $P \leqslant \mu^2TT^*$ for some $\mu \geqslant 0$. Since $\operatorname{Dom}((TT^*)^{1/2}) = \operatorname{Dom}(T^*)$, this means that for arbitrary $y \in \operatorname{Dom}(T^*)$ we should have $\|Py\| \leqslant \mu\|T^*y\|$, that is,

$$\left|\langle x, y\rangle_{\mathcal{H}}\right|\frac{\|x\|}{\|x\|^2} \leqslant \mu\left\|T^*y\right\|.$$

3. Closely embedded Hilbert spaces

3.1. Definition and some properties

As a consequence of the results presented in the previous section a natural generalization of the notion of continuously embedded Hilbert spaces can be singled out.

Let $\mathcal{H}$ and $\mathcal{H}_+$ be two Hilbert spaces. The Hilbert space $\mathcal{H}_+$ is called *closely embedded* in $\mathcal{H}$ if:

(ceh1) There exists a linear manifold $\mathcal{D} \subseteq \mathcal{H}_+ \cap \mathcal{H}$ that is dense in $\mathcal{H}_+$.
(ceh2) The embedding operator j_+ with domain $\mathcal{D}$ is closed, as an operator $\mathcal{H}_+ \to \mathcal{H}$.

In order to avoid possible confusions, let us point out that the meaning of the axiom (ceh1) is that on $\mathcal{D}$ the algebraic structures of $\mathcal{H}_+$ and $\mathcal{H}$ agree. Also, recall that in case $\mathcal{H}_+ \subseteq \mathcal{H}$ and the embedding operator $j_+ : \mathcal{H}_+ \to \mathcal{H}$ is continuous, one says that $\mathcal{H}_+$ is *continuously embedded* in $\mathcal{H}$, e.g. see P.A. Fillmore and J.P. Williams [15] and the bibliography cited there.

Let us observe that the above definition is consistent with the model $\mathcal{R}(T)$, for $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$, more precisely, if $\mathcal{H}_+$ is closely embedded in $\mathcal{H}$ then $\mathcal{R}(j_+) = \mathcal{H}_+$ and $\|x\|_+ = \|x\|_{j_+}$.

We first show that closely embedded Hilbert spaces provide yet another representation of Hilbert spaces induced by nonnegative selfadjoint operators.

Proposition 3.1. *Let $\mathcal{H}_+$ be a Hilbert space closely embedded in $\mathcal{H}$. Then $(\mathcal{H}_+; j_+^*)$ is a Hilbert space induced by $A = j_+ j_+^*$.*

Proof. Since j_+ is closed and densely defined it follows that the operator j_+^* is densely defined (and closed). The operator $A = j_+ j_+^*$ is a positive selfadjoint operator in $\mathcal{H}$ and $\operatorname{Dom}(j_+^*)$ is a core of A. In addition, since j_+ is injective, it follows that j_+^* has the range dense in $\mathcal{H}_+$ and hence

$$\mathcal{H}_+ = \overline{\operatorname{Ran}(j_+^*)} = \overline{j_+^*(\operatorname{Dom}(j_+ j_+^*))} = \overline{j_+^*(\operatorname{Dom}(A))}.$$

Thus, the first three axioms from the definition of the induced Hilbert space are verified. The fourth axiom is clear since $A = j_+ j_+^*$. □

Following L. Schwartz [28], we call $A = j_+ j_+^*$ the *kernel operator* of the closely embedded Hilbert space $\mathcal{H}_+$ with respect to $\mathcal{H}$.

The model for closely embedded Hilbert spaces follows the results on the Hilbert space $\mathcal{R}(T)$ as presented in Section 2.2. Thus, if $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ then the Hilbert space $\mathcal{R}(T)$, with its canonical embedding j_T as defined in (2.3) and (2.4), is a Hilbert space closely contained in $\mathcal{H}$, e.g. by Proposition 2.7. Conversely, if $\mathcal{H}_+$ is a Hilbert space closely contained in $\mathcal{H}$, and j_+ denotes its canonical closed embedding, then $\mathcal{H}_+$ can be naturally viewed as the Hilbert space of type $\mathcal{R}(j_+)$. This fact is actually more general.

Proposition 3.2. *Let $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ and consider the Hilbert space $\mathcal{R}(T)$ closely contained in $\mathcal{H}$, with its canonical closed embedding j_T. Then $TT^* = j_T j_T^*$.*

Proof. Since T is closed and densely defined, we can apply Lemma 2.5 and write $T = j_T U_T$, where $U_T \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is a coisometry with $\operatorname{Ker}(U_T) = \operatorname{Ker}(T)$. Then $T^* = U_T^* j_T^*$ and $TT^* = j_T U_T U_T^* j_T^* = j_T j_T^*$, where the equalities in the sense of unbounded operators can be easily verified. □

Since closely embedded Hilbert spaces are special cases of induced Hilbert spaces, we can reformulate the lifting theorem. Thus, from Proposition 3.1 and Theorem 2.2 we get:

Theorem 3.3. *Let A and B be nonnegative selfadjoint operators in the Hilbert spaces $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, and let $\mathcal{K}_A$ and $\mathcal{K}_B$ be Hilbert spaces closely embedded in $\mathcal{H}_1$ and, respectively $\mathcal{H}_2$, with kernel operators A and, respectively, B. For any operators $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle Bx, Ty \rangle_{\mathcal{H}_2} = \langle Sx, Ay \rangle_{\mathcal{H}_1}, \quad x \in \operatorname{Dom}(B),\ y \in \operatorname{Dom}(A), \tag{3.1}$$

there exist uniquely determined operators $\widetilde{T} \in \mathcal{B}(\mathcal{K}_A, \mathcal{K}_B)$ and $\widetilde{S} \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_A)$ such that $\widetilde{T} j_A^* x = j_B^* T x$ for all $x \in \operatorname{Dom}(A)$, $\widetilde{S} j_B^* y = j_A^* S y$ for all $y \in \operatorname{Dom}(B)$, and

$$\langle \widetilde{S} h, k \rangle_{\mathcal{K}_A} = \langle h, \widetilde{T} k \rangle_{\mathcal{K}_B}, \quad h \in \mathcal{K}_B,\ k \in \mathcal{K}_A. \tag{3.2}$$

3.2. Uniqueness

As in the case of bounded kernel operators and continuous embeddings, we can prove that Hilbert spaces that are closely embedded in a given Hilbert space are uniquely determined by their kernel operators. As expected, the uniqueness takes a slightly weaker form.

Theorem 3.4. *Let $\mathcal{H}_+$ be a Hilbert space closely embedded in $\mathcal{H}$, with $j_+ : \mathcal{H}_+ \to \mathcal{H}$ its densely defined and closed embedding operator, and let $A = j_+ j_+^*$ be the kernel operator of $\mathcal{H}_+$. Then:*

(a) $\mathrm{Ran}(A^{1/2}) = \mathrm{Dom}(j_+)$ *is dense in both* $\mathcal{R}(A^{1/2})$ *and* $\mathcal{H}_+$.
(b) *For all* $x \in \mathrm{Ran}(A^{1/2})$ *and all* $y \in \mathrm{Dom}(A)$ *we have* $\langle x, y\rangle_{\mathcal{H}} = \langle x, Ay\rangle_+ = \langle x, Ay\rangle_{A^{1/2}}$.
(c) $\mathrm{Ran}(A)$ *is dense in both* $\mathcal{R}(A^{1/2})$ *and* $\mathcal{H}_+$.
(d) *For any* $x \in \mathrm{Dom}(j_+) = \mathrm{Ran}(A^{1/2})$ *we have*

$$\|x\|_+ = \sup\left\{ \frac{|\langle x, y\rangle_{\mathcal{H}}|}{\|A^{1/2}y\|_{\mathcal{H}}} \;\middle|\; y \in \mathrm{Dom}\bigl(A^{1/2}\bigr),\ A^{1/2}y \neq 0 \right\}.$$

(e) *The identity operator* : $\mathrm{Ran}(A) (\subseteq \mathcal{R}(A^{1/2})) \to \mathcal{H}_+$ *uniquely extends to a unitary operator* $V : \mathcal{R}(A^{1/2}) \to \mathcal{H}_+$ *such that* $VAx = j_+^* x$, *for all* $x \in \mathrm{Dom}(A)$.

Proof. (a) Let $j_+^* = W|j_+^*|$ be the polar decomposition of the closed operator j_+^*. Then $j_+ = |j_+^*|W^*$, $|j_+^*| = (j_+ j_+^*)^{1/2} = A^{1/2}$, and hence $\mathrm{Dom}(j_+) = \mathrm{Ran}(j_+) = \mathrm{Ran}(A^{1/2})$ is dense in both $\mathcal{R}(A^{1/2})$ and $\mathcal{H}_+$, by definition.

(b) Let $x \in \mathrm{Ran}(A^{1/2}) = \mathrm{Dom}(j_+)$ and $y \in \mathrm{Dom}(A)$ be arbitrary. Then $\langle x, y\rangle_{\mathcal{H}} = \langle j_+ x, y\rangle_{\mathcal{H}} = \langle x, j_+^* y\rangle_+ = \langle x, Ay\rangle_+$, by Proposition 3.1. On the other hand, by representing $x = A^{1/2}u$ for some $u \in \mathrm{Dom}(A^{1/2})$ and $u \perp \mathrm{Ker}(A^{1/2})$, we have $\langle x, Ay\rangle_{A^{1/2}} = \langle A^{1/2}u, Ay\rangle_{A^{1/2}} = \langle u, A^{1/2}y\rangle_{\mathcal{H}} = \langle A^{1/2}u, y\rangle_{\mathcal{H}} = \langle x, y\rangle_{\mathcal{H}}$.

(c) Let $u \in \mathcal{R}(A^{1/2})$ and let $\langle u, Ay\rangle_{A^{1/2}} = 0$ for $y \in \mathrm{Dom}(A)$. Since $u \in \mathcal{R}(A^{1/2})$ we can find a sequence (x_n) of elements $x_n \in \mathrm{Dom}(A^{1/2})$ such that $x_n \perp \mathrm{Ker}(A^{1/2})$, $A^{1/2}x_n \to u$ in $\mathcal{R}(A^{1/2})$ and $x_n \to x$ in $\mathcal{H}$. Obviously $x \perp \mathrm{Ker}(A^{1/2})$. We have

$$0 = \langle u, Ay\rangle_{A^{1/2}} = \lim_{n\to\infty} \bigl\langle A^{1/2}x_n, \bigl(A^{1/2}\bigr)^2 y\bigr\rangle_{A^{1/2}} = \lim_{n\to\infty} \bigl\langle x_n, A^{1/2}y\bigr\rangle_{\mathcal{H}} = \bigl\langle x, A^{1/2}y\bigr\rangle_{\mathcal{H}}$$

for any $y \in \mathrm{Dom}(A)$. Thus, $x \perp \overline{\mathrm{Ran}(A^{1/2})}$ and hence $x \in \mathrm{Ker}(A^{1/2})$. Consequently, $x = 0$.

Then

$$\|u\|_{A^{1/2}}^2 = \lim_{n\to\infty} \bigl\|A^{1/2}x_n\bigr\|_{A^{1/2}}^2 = \lim_{n\to\infty} \bigl\langle A^{1/2}x_n, A^{1/2}x_n\bigr\rangle_{A^{1/2}} = \lim_{n\to\infty} \|x_n\|_{\mathcal{H}} = 0,$$

and hence $u = 0$.

(d) As a consequence of (b) and (c) the inner products $\langle\cdot,\cdot\rangle_+$ and $\langle\cdot,\cdot\rangle_{A^{1/2}}$ coincide on $\mathrm{Ran}(A^{1/2}) = \mathrm{Dom}(j_+)$, and the same holds for their norms $\|\cdot\|_+$ and $\|\cdot\|_{A^{1/2}}$. Thus, the required formula for the norm $\|\cdot\|_+$ follows from Theorem 2.10.

(e) Again, this follows from the fact that, on $\mathrm{Ran}(A^{1/2}) = \mathrm{Dom}(j_+)$ the two norms $\|\cdot\|_+$ and $\|\cdot\|_{A^{1/2}}$ coincide, and by the assertion (a). □

As pointed out before, any closely embedded Hilbert space is, in an "essentially unique" way, a Hilbert space of type $\mathcal{R}(T)$. We can now clarify even more the meaning of this "essentially unique" feature.

Corollary 3.5. *Let $T_i \in \mathcal{C}(\mathcal{G}_i, \mathcal{H})$ and consider the associated Hilbert spaces $\mathcal{R}(T_i)$ as well as their closed embeddings $j_i = j_{T_i}$, for $i = 1, 2$. The following assertions are equivalent:*

(i) $T_1 T_1^* = T_2 T_2^*$.
(ii) $T_1 = T_2 V$ *for some partial isometry* $V \in \mathcal{B}(\mathcal{G}_1, \mathcal{G}_2)$ *with* $\mathrm{Ker}(V) = \mathrm{Ker}(T_1)$ *and* $\mathrm{Ker}(V^*) = \mathrm{Ker}(T_2)$.
(iii) $\mathrm{Dom}(j_1) = \mathrm{Dom}(j_2)$ *and* $\|x\|_1 = \|x\|_2$ *for all x in this common domain.*

3.3. Hilbert spaces of holomorphic functions in the unit disk

In this subsection we exemplify the results on closely embedded Hilbert spaces to some Hilbert space of holomorphic functions in the unit disk.

Let $H^2(\mathbb{D})$ be *the Hardy space* of holomorphic functions on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and such that

$$\|f\|_2^2 = \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} \bigl|f\bigl(re^{it}\bigr)\bigr|^2 \,\mathrm{d}t < \infty.$$

Let $\mathcal{D}$ be *the Dirichlet space* of functions f holomorphic in the open unit disk $\mathbb{D}$ and such that

$$\|f\|_0^2 = \left|f(0)\right|^2 + \frac{1}{\pi}\int_{\mathbb{D}} \left|f'(z)\right|^2 \mathrm{d}(x,y) < \infty, \quad z = x + \mathrm{i}y,$$

and finally, let $A^2(\mathbb{D})$ be *the Bergman space* of functions f holomorphic in the open unit disk $\mathbb{D}$ and such that

$$\|f\|_1^2 = \frac{1}{\pi}\int_{\mathbb{D}} \left|f(z)\right|^2 \mathrm{d}(x,y) < \infty, \quad z = x + \mathrm{i}y.$$

Proposition 3.6.

(1) *The Hardy space $H^2(\mathbb{D})$ is closely, but not continuously, embedded in the Dirichlet space $\mathcal{D}$, more precisely, the closed embedding is j_0, with domain* $\mathrm{Dom}(j_0) = \mathcal{D}$ *considered as a linear manifold in $H^2(\mathbb{D})$, and the kernel operator K with maximal domain*

$$(Kf)(z) = zf'(z) + f(0), \quad f \in \mathrm{Dom}(K) := \{f \in \mathcal{D} \mid Kf \in \mathcal{D}\}.$$

(2) *The Bergman space $A^2(\mathbb{D})$ is closely, but not continuously, embedded in the Hardy space $H^2(\mathbb{D})$, more precisely, the closed embedding is j_1, with domain* $\mathrm{Dom}(j_1) = H^2(\mathbb{D})$ *considered as a linear manifold in $A^2(\mathbb{D})$, and the kernel operator A with maximal domain*

$$(Af)(z) = \big(zf(z)\big)' = f(z) + zf'(z), \quad f \in \mathrm{Dom}(A) := \big\{f \in H^2(\mathbb{D}) \bigm| Af \in H^2(\mathbb{D})\big\}.$$

(3) *The Bergman space $A^2(\mathbb{D})$ is closely, but not continuously, embedded in the Dirichlet space $\mathcal{D}$, more precisely, the closed embedding is j, with domain* $\mathrm{Dom}(j) = \mathcal{D}$ *considered as a linear manifold of $A^2(\mathbb{D})$, and the kernel operator B with maximal domain*

$$(Bf)(z) = z\big(z^2 f'(z)\big)' + f(0), \quad \mathrm{Dom}(B) := \big\{f \in A^2(\mathbb{D}) \bigm| Bf \in A^2(\mathbb{D})\big\}.$$

In order to prove this proposition we first prove an abstract result on closed embeddings of weighted ℓ^2 spaces, that may be interesting by itself as well. For $w = (w_n)$ an arbitrary sequence of strictly positive numbers, let ℓ^2_w be the Hilbert space of weighted square summable complex sequences

$$\ell^2_w = \left\{x = (\xi_n) \Bigm| \sum_n w_n|\xi_n|^2 < \infty\right\}, \tag{3.3}$$

with inner product

$$\langle x, y\rangle_w = \sum_n w_n \xi_n \overline{\eta}_n, \tag{3.4}$$

where $x = (\xi_n)$ and $y = (\eta_n)$ are both in ℓ^2_w.

In the following we use the notation $wx = (w_n x_n)$ for coordinatewise multiplication of numerical sequences.

Proposition 3.7. *Let $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ be two sequences of strictly positive real numbers, and let $w_n = \beta_n/\alpha_n$.*

If $\sup_n w_n = \infty$ then ℓ^2_α is closely, but not continuously, embedded in ℓ^2_β, more precisely, the embedding operator j_w with the domain $\mathrm{Dom}(j_w) = \ell^2_\alpha \cap \ell^2_\beta$ *is closed, as an operator from ℓ^2_α into ℓ^2_β, and the kernel operator is the operator of multiplication with w*

$$M_w x = wx, \quad x \in \mathrm{Dom}(M_w)$$

defined in the space ℓ^2_β on the domain

$$\mathrm{Dom}(M_w) = \big\{x \in \ell^2_\beta \bigm| wx \in \ell^2_\alpha \cap \ell^2_\beta\big\}.$$

If $\sup_n w_n < \infty$, then ℓ^2_α is continuously embedded in ℓ^2_β, and the embedding operator j_w has the bound $(\sup_n |w|_n)^{1/2}$.

Proof. Assume that $\sup_n w_n = +\infty$. Observe that $\mathrm{Dom}(j_w) = \ell^2_\alpha \cap \ell^2_\beta$ is dense in ℓ^2_α since it contains all finite sequences, and it is a simple exercise to prove that the identity $j_w : \mathrm{Dom}(j_w)(\subseteq \ell^2_\alpha) \to \ell^2_\beta$ is a closed operator. Thus, its adjoint j_w^* is a densely defined closed operator as well, and for all $x = (\xi_n) \in \ell^2_\alpha \cap \ell^2_\beta$ and all $y = (\eta_n) \in \mathrm{Dom}(j_w^*)$ we have

$$\langle j_w x, y\rangle_\beta = \sum_n \beta_n \xi_n \overline{\eta}_n = \sum_n \alpha_n \xi(\overline{w_n \eta_n}) = \langle x, wy\rangle_\alpha.$$

This calculation shows that

$$\mathrm{Dom}\big(j_w^*\big) = \big\{y \in \ell^2_\beta \bigm| wy \in \ell^2_\alpha\big\},$$

and that

$$j_w^* y = wy, \quad y \in \operatorname{Dom}(j_w^*).$$

Therefore, the kernel operator $M_w = j_w j_w^*$ is the operator of multiplication with w defined on vectors $y \in \ell_\beta^2$ such that $wy \in \ell_\alpha^2 \cap \ell_\beta^2$.

In case $\sup_n w_n < \infty$ we have

$$\|x\|_\beta^2 = \sum_n \beta_n |\xi_n|^2 = \sum_n \alpha_n w_n |\xi_n| \leqslant \Bigl(\sup_n w_n\Bigr) \|x\|_\alpha^2$$

for any $x = (\xi_n) \in \ell_\alpha^2$, and thus ℓ_α^2 is continuously embedded in ℓ_β^2, and the bound of the embedding operator j_w is $(\sup_n w_n)^{1/2}$. □

Proof of Proposition 3.6. (1) If $f(z) = \sum_{n\geqslant 0} a_n z^n$ is the Taylor expansion of an arbitrary function f holomorphic in the unit disk $\mathbb{D}$ then

$$\|f\|_2^2 = \sum_{n\geqslant 0} a_n^2, \qquad \|f\|_0^2 = |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2,$$

which provides the usual isometric isomorphisms between the Hardy space $H^2(\mathbb{D})$ and ℓ^2, and respectively, between the Dirichlet space $\mathcal{D}$ and ℓ_w^2 with the weight $w = (1, 1, 2, 3, \ldots, n, n+1, \ldots)$. Since $\sup_n w_n = +\infty$ we can apply Proposition 3.7. If $f(z) = \sum_{n\geqslant 0} a_n z^n$ is the Taylor expansion of an arbitrary function f holomorphic in the unit disk $\mathbb{D}$, identified with the sequence $x = (a_n)_{n\geqslant 0}$, then

$$zf'(z) + f(0) = a_0 + \sum_{n\geqslant 1} n a_n z^n,$$

is identified with the sequence $wx = (a_0, a_1, 2a_2, 3a_3, \ldots, na_n, (n+1)a_{n+1}, \ldots)$, and from here it follows easily that the operator K is the kernel operator of the closed embedding of the Hardy space $H^2(\mathbb{D})$ into the Dirichlet space $\mathcal{D}$.

(2) If $f(z) = \sum_{n\geqslant 0} a_n z^n$ is the Taylor expansion of an arbitrary function f holomorphic in the unit disk $\mathbb{D}$ then

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}, \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

hence, the Bergman space $A^2(\mathbb{D})$ is actually a Hilbert space of type ℓ_w^2 with $w = (\frac{1}{n+1})$. Therefore, since $\inf w_n = 0$, again by Proposition 3.7, it follows that the Bergman space $A^2(\mathbb{D})$ is closely (but not continuously) embedded in the Hardy space $H^2(\mathbb{D})$. The formula for the kernel operator A follows from the calculation

$$\bigl(zf(z)\bigr)' = \sum_{n\geqslant 0} (n+1) a_n z^n$$

for $f(z) = \sum_{n\geqslant 0} a_n z^n$ identified with the sequence $x = (a_n)$, showing that $(zf(z))'$ is identified with the sequence $w^{-1}x = ((n+1)a_n)$.

(3) We use the unitary identification of $\mathcal{D}$ with ℓ_α where $\alpha_n = 1/(n+1)$, and of $A^2(\mathbb{D})$ with ℓ_β where $\beta_n = n$ for $n \geqslant 1$ and $\beta_0 = 1$, and then Proposition 3.7. The formula for the kernel operator B follows from the calculation

$$z\bigl(z^2 f'(z)\bigr)' + f(0) = a_0 + \sum_{n\geqslant 1} n(n+1) a_n z^n$$

whenever $f(z) = \sum_{n\geqslant 0} a_n z^n$ is the Taylor expansion of a function f holomorphic in $\mathbb{D}$. □

4. Closed embeddings of homogeneous Sobolev spaces

In this section we present a first application of our results to some homogeneous Sobolev spaces. Throughout this section we will use the basic notation and facts on Sobolev spaces as in R.A. Adams [1] and V.G. Maz'ja [24].

4.1. A homogeneous Sobolev space on $\mathbb{R}_+$

We consider the Lebesgue space $L_2(\mathbb{R}_+)$ and the *homogeneous Sobolev space* $H_2^1(\mathbb{R}_+)$ of all absolutely continuous functions u on each bounded interval of the nonnegative semi-axis and $u(0) = 0$, such that

$$\|u\|_{1,2}^2 := \int_0^{\infty} \bigl(\bigl|t^{-1}u(t)\bigr|^2 + \bigl|u'(t)\bigr|^2\bigr)\, \mathrm{d}t < \infty.$$

Theorem 4.1. *The homogeneous Sobolev space $H_2^1(\mathbb{R}_+)$ is closely, but not continuously, embedded in the Lebesgue space $L_2(\mathbb{R}_+)$, more precisely, the embedding j has domain $\mathrm{Dom}(j) = \overset{\circ}{W}{}_2^1(\mathbb{R}_+)$, the Sobolev space of all functions $u \in L_2(\mathbb{R}_+)$ such that u is absolutely continuous on each bounded interval in $\mathbb{R}_+$, $u' \in L_2(\mathbb{R}_+)$ (in the sense of distributions), and $u(0) = 0$, and having as kernel operator the integral operator K with kernel*

$$k(t,s) = t\chi_{[t,\infty)}(s) + s\chi_{[0,t)}(s) \quad (0 < t, s < \infty),$$

defined in $L_2(\mathbb{R}_+)$ on its maximal domain.

Proof. Let T be the operator of integration in $L_2(\mathbb{R}_+)$, that is,

$$(Tu)(t) = \int_0^t u(s)\,\mathrm{d}s, \quad u \in \mathrm{Dom}(T)$$

defined on its maximal domain $\mathrm{Dom}(T) = \{u \in L_2(\mathbb{R}_+) \mid Tu \in L_2(\mathbb{R}_+)\}$. The operator T is injective, densely defined, and closed. Its inverse T^{-1} is the derivation operator $T^{-1}u = u'$ in $L_2(\mathbb{R}_+)$, with $\mathrm{Dom}(T^{-1}) = \mathrm{Ran}(T)$ coinciding with the Sobolev space $\overset{\circ}{W}{}_2^1(\mathbb{R}_+)$. With the definitions as in (2.3) on $\mathrm{Ran}(T)$ we have

$$\langle u, v\rangle_T = \int_0^\infty u'(t)\overline{v'(t)}\,\mathrm{d}t, \quad u, v \in \overset{\circ}{W}{}_2^1(\mathbb{R}_+),$$

and, respectively, the *Dirichlet norm*

$$\|u\|_T = \|u'\|_{L_2(\mathbb{R}_+)}, \quad u \in \overset{\circ}{W}{}_2^1(\mathbb{R}_+).$$

By means of the Hardy inequality (see [18, Theorem 327, p. 240])

$$\int_0^\infty \left| t^{-1}\int_0^t u(s)\,\mathrm{d}s\right|^2 \mathrm{d}t \leqslant 4\int_0^\infty |u(t)|^2\,\mathrm{d}t, \quad u \in \mathrm{Dom}(T),$$

which, denoting $v(t) = \int_0^t u(s)\,\mathrm{d}s$, can be equivalently rewritten as

$$\int_0^\infty |t^{-1}v(t)|^2\,\mathrm{d}t \leqslant 4\int_0^\infty |v'(t)|^2\,\mathrm{d}t, \quad v \in \mathrm{Ran}(T),$$

it is seen that on $\mathrm{Ran}(T)$ $(=\overset{\circ}{W}{}_2^1(\mathbb{R}_+))$ the Dirichlet norm is equivalent with the following one

$$\|u\|_{2,1}^2 := \int_0^\infty \left(|t^{-1}u(t)|^2 + |u'(t)|^2\right)\mathrm{d}t, \quad u \in \overset{\circ}{W}{}_2^1(\mathbb{R}_+).$$

Taking into account these facts we conclude that the completion $\mathcal{R}(T)$ of the pre-Hilbert space $(\mathrm{Ran}(T); \langle\cdot,\cdot\rangle_T)$ coincides with the homogeneous Sobolev space $H_2^1(\mathbb{R}_+)$. By Proposition 3.2 the space $\mathcal{R}(T)$ is closely embedded (but not continuously) in $L_2(\mathbb{R}_+)$, and the kernel of the embedding operator is the integral operator

$$\begin{aligned}(TT^*u)(t) &= \int_0^t \left(\int_s^\infty u(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s = t\int_t^\infty u(s)\,\mathrm{d}s + \int_0^t su(s)\,\mathrm{d}s \\ &= \int_0^\infty \left(t\chi_{[t,\infty)}(s) + s\chi_{[0,t)}(s)\right)u(s)\,\mathrm{d}s\end{aligned}$$

defined in $L_2(\mathbb{R}_+)$ on its maximal domain. □

Remark 4.2. Consider the operator of integration on a finite interval, for instance, let

$$(Tu)(t) = \int_0^t u(s)\,\mathrm{d}s \quad (0 < t < 1).$$

In this case T represents a bounded operator on the space $L_2[0,1]$, its range $\mathrm{Ran}(T)$ coincides with the Sobolev space $\overset{\circ}{W}{}^1_2[0,1]$ consisting of all absolutely continuous functions u on $[0,1]$ such that $u' \in L_2[0,1]$ and $u(0)=0$. We have $\mathcal{R}(T) = \overset{\circ}{W}{}^1_2[0,1]$, the canonical embedding operator j_T is bounded (even compact) having as kernel operator the integral operator

$$(Ku)(t) = t\int_0^1 u(s)\,\mathrm{d}s + \int_0^t su(s)\,\mathrm{d}s, \quad u \in L_2[0,1].$$

Remark 4.3. Theorem 3.4 leaves unanswered the question on how large the overlapping of a closely embedded Hilbert space $\mathcal{H}_+$ and its ambiental Hilbert space $\mathcal{H}$ can be. The minimal possibility is $\mathrm{Dom}(j_+)$ but, as the following examples show, in general, this overlapping can be minimal or not.

(a) We consider $\mathcal{H}_+$ the Hilbert space closely contained in $\mathcal{H}$ as in Proposition 3.6 or Proposition 3.7. In this case $\mathcal{H}\cap\mathcal{H}_+$ coincides with the domain of the corresponding closed embedding, and hence it is minimal.

(b) We now consider $\mathcal{H}_+$ the Hilbert space closely contained in $\mathcal{H} = L_2(\mathbb{R}_+)$ as in Theorem 4.1. Let α be a number in the interval $(1/2, 1)$ and consider the function

$$u(t) = \frac{1}{1+t^\alpha}. \tag{4.1}$$

Then

$$u'(t) = -\frac{\alpha t^{\alpha-1}}{(1+t^\alpha)^2} = -\frac{\alpha}{t^{1-\alpha}(1+t^\alpha)^2}.$$

Note that $u, u' \in L_2(\mathbb{R}_+)$ but $u \notin \mathrm{Ran}(T)$, hence the overlapping between $\mathcal{H}_+$ and $\mathcal{H}$ is not minimal, in this case.

4.2. A homogeneous Sobolev space on $\mathbb{R}^n$

Let $\mathcal{H} = L_2(\mathbb{R}^n)$ and, for $2l < n$, let $H^l_2(\mathbb{R}^n)$ denote the homogeneous Sobolev space of all functions $u \in W^l_{2,\mathrm{loc}}(\mathbb{R}^n)$ for which $\|u\|^2_{2,l} < \infty$, where

$$\|u\|^2_{2,l} := \int_{\mathbb{R}^n} \left(\left|(\nabla_l u)(x)\right|^2 + |x|^{-2l}\left|u(x)\right|^2\right)\mathrm{d}x, \quad u \in C_0^\infty\left(\mathbb{R}^n\right). \tag{4.2}$$

Theorem 4.4. *The homogeneous Sobolev space $H^l_2(\mathbb{R}^n)$, with $2l < n$, is closely, but not continuously, embedded in the Lebesgue space $L_2(\mathbb{R}^n)$, more precisely, the embedding j has domain the Sobolev space $W^l_2(\mathbb{R}^n)$, and kernel operator the M. Riesz potential $I_\alpha = (-\Delta)^{-\alpha/2}$ of order $\alpha = 2\ell$.*

Proof. Let $H = (-\Delta)^l$ defined on its maximal domain, i.e. on the Sobolev space $W^\alpha_2(\mathbb{R}^n)$, $\alpha = 2l$. Below we always assume that $2l < n$. H represents a selfadjoint operator in $\mathcal{H}$.

Next, we consider the operator T defined in the space $L_2(\mathbb{R}^n)$ by

$$(Tu)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\xi|^{-l/2}\widehat{u}(\xi)e^{-\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^n,$$

on the domain

$$\mathrm{Dom}(T) := \left\{u \in L_2\left(\mathbb{R}^n\right) \,\middle|\, |\xi|^{-l/2}\widehat{u}(\xi) \in L_2\left(\mathbb{R}^n\right)\right\}.$$

The operator T can be written formally as

$$T = (-\Delta)^{-l/2},$$

and it can be also considered as *the M. Riesz potential of order l*, e.g. see E.M. Stein [30, § V.1.1], that means that T is the convolution integral operator with the kernel $|x|^{l-n}$, up to a multiplicative constant,

$$(Tu)(x) = c\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-l}}\,\mathrm{d}y, \quad u \in \mathrm{Dom}(T).$$

T represents a closed unbounded operator in $\mathcal{H}$ $(= L_2(\mathbb{R}^n))$, and, obviously, $\mathrm{Ker}(T) = \{0\}$. The domain of T is $\mathrm{Ran}(H^{1/2})$ and its range is $\mathrm{Dom}(H^{1/2})$, i.e. the Sobolev space $W^l_2(\mathbb{R}^n)$.

According to (2.3) we define an inner product on $\mathrm{Ran}(T)$ by

$$\langle Tf, Tg\rangle_T := \langle f, g\rangle_{\mathcal{H}}, \quad f, g \in \mathrm{Dom}(T). \tag{4.3}$$

Thus, for $u, v \in \mathrm{Ran}(T)$ $(= W_2^l(\mathbb{R}^n))$, we have

$$\langle u, v\rangle_T = \big\langle T^{-1}u, T^{-1}v\big\rangle_{\mathcal{H}} = \big\langle (-\Delta)^{l/2}u, (-\Delta)^{l/2}v\big\rangle_{\mathcal{H}}, \tag{4.4}$$

and, respectively, for the corresponding norm of $u \in \mathrm{Ran}(T)$,

$$\|u\|_T = \big\|(-\Delta)^{l/2}u\big\|_{\mathcal{H}}, \quad u \in \mathrm{Ran}(T). \tag{4.5}$$

Thus, the norm on $\mathrm{Ran}(T)$ corresponding to the inner product defined by (4.3) is in fact the Dirichlet norm (4.5). $\mathrm{Ran}(T)$ $(= W_2^l(\mathbb{R}^n))$ endowed with the norm (4.5) is not a complete space. The corresponding completion $\mathcal{R}(T)$ can be described by using the following Hardy type inequality, e.g. as in V.G. Maz'ja [24, § 2.1.6] (see also E.B. Davies [10] for a recent review)

$$\int_{\mathbb{R}^n} |x|^{-2l}\big|u(x)\big|^2\,\mathrm{d}x \leqslant c\int_{\mathbb{R}^n}\big|(\nabla_l u)(x)\big|^2\,\mathrm{d}x, \quad 2l < n,\ u \in C_0^\infty\big(\mathbb{R}^n\big), \tag{4.6}$$

where, by definition

$$\int_{\mathbb{R}^n}\big|(\nabla_l u)(x)\big|^2\,\mathrm{d}x = \int_{\mathbb{R}^n}|\xi|^{2l}\big|\widehat{u}(\xi)\big|^2\,\mathrm{d}\xi, \quad u \in W_{2,\mathrm{loc}}^l\big(\mathbb{R}^n\big)$$

($\widehat{u}$ is the Fourier transform of u).

By the inequality (4.6) it follows that on $C_0^\infty(\mathbb{R}^n)$ the norms defined by (4.5) and (4.2), respectively, are equivalent. Therefore, the completion $\mathcal{R}(T)$ of the pre-Hilbert space $(\mathrm{Ran}(T); \langle\cdot,\cdot\rangle_T)$ (note that $C_0^\infty(\mathbb{R}^n)$ is dense in $W_2^l(\mathbb{R}^n)$ $(= \mathrm{Ran}(T))$) coincides algebraically and topologically with the homogeneous Sobolev space $H_2^l(\mathbb{R}^n)$. $\mathcal{R}(T)$ is closely (not continuously) embedded in the space $\mathcal{H}$ $(= L_2(\mathbb{R}_+))$ (cf. Proposition 2.7). Moreover, from (4.4) we have

$$\langle u, v\rangle_T = \langle Hu, v\rangle_{\mathcal{H}}, \quad u \in \mathrm{Dom}(H),\ v \in \mathrm{Ran}(T). \qquad \square$$

4.3. The Bessel potential

The proof of Theorem 4.4 was based on a certain factorization that used the singular integral operator associated to the Riesz potential. In this section we show that, if the Bessel potential is used instead, we get a continuously embedded Sobolev space.

Let $\mathcal{H} = L_2(\mathbb{R}^n)$, $n \geqslant 3$, and let H denote the operator

$$H = (-\Delta + I)^l,$$

where $\Delta \equiv \sum_{k=1}^n \partial^2/\partial x_k^2$ is the Laplacian, l is a positive number (not necessary an integer). As the domain of H the Sobolev space $W_2^\alpha(\mathbb{R}^n)$, $\alpha = 2l$, is considered. H represents on this domain a positive definite selfadjoint operator. In particular, H is an invertible operator, and its inverse is bounded on $\mathcal{H}$. Next, we denote

$$T = (-\Delta + I)^{-l/2}.$$

The operator T can be represented, e.g. see E.M. Stein [30, § V.3.1], as a convolution integral operator with kernel

$$G(x) = cK_{(n-l)/2}\big(|x|\big)|x|^{(l-n)/2},$$

where K_ν is the modified Bessel function of the third kind, where c is a positive constant, see e.g. N. Aronszajn and K.T. Smith [6, § II.3]. Thus

$$(Tu)(x) = \int_{\mathbb{R}^n} G_l(x-y)u(y)\,\mathrm{d}y, \quad u \in L_2\big(\mathbb{R}^n\big).$$

This integral operator is known as *the Bessel potential of order l*, e.g. see [30].

Note that T can be also regarded as a pseudodifferential operator corresponding to the symbol $(1+|\xi|^2)^{-l/2}$, i.e.

$$(Tu)(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\big(1+|\xi|^2\big)^{-l/2}\,\widehat{u}(\xi)e^{-\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^n,$$

where $\widehat{u} = Fu$ is the Fourier transform of the function $u \in L_2(\mathbb{R}^n)$ ($\langle x, \xi\rangle$ denotes the scalar product of the elements $x, \xi \in \mathbb{R}^n$). Obviously, T maps $L_2(\mathbb{R}^n)$ onto $W_2^l(\mathbb{R}^n)$.

According to (2.3) we define an inner product on $\operatorname{Ran}(T)$ $(= W_2^l(\mathbb{R}^n))$ by setting

$$\langle Tf, Tg\rangle_T := \langle f, g\rangle_{\mathcal{H}}, \quad f, g \in \mathcal{H}.$$

We have

$$\langle u, v\rangle_T = \big\langle (-\Delta + I)^{l/2}u, (-\Delta + I)^{l/2}v\big\rangle_{\mathcal{H}}, \quad u, v \in \operatorname{Ran}(T),$$

and, respectively, for the corresponding norm

$$\|u\|_T = \big\|(-\Delta + I)^{l/2}u\big\|_{\mathcal{H}}, \quad u \in \operatorname{Ran}(T).$$

This norm is equivalent with the standard norm

$$\|u\|_{W_2^l(\mathbb{R}^n)} := \bigg(\int_{\mathbb{R}^n} \big|u(x)\big|^2\,\mathrm{d}x + \int_{\mathbb{R}^n} \big|(\nabla u)(x)\big|^2\,\mathrm{d}x\bigg)^{1/2}$$

of the Sobolev space $W_2^l(\mathbb{R}^n)$. Consequently, $\operatorname{Ran}(T)$ endowed with the norm $\|\cdot\|_T$ coincides with the Sobolev space $W_2^l(\mathbb{R}^n)$. Thus $\mathcal{R}(T) = W_2^l(\mathbb{R}^n)$ algebraically and topologically (cf. Remark 2.9(a)). Moreover, $\mathcal{R}(T)$ is continuously embedded in $\mathcal{H}$ (cf Remark 2.9(b)), and the kernel operator of the canonical embedding is the Bessel potential $J_\alpha = (-\Delta + I)^{-\alpha/2}$ of order $\alpha = 2\ell$. Note that

$$\langle u, v\rangle_T = \langle Hu, v\rangle_{\mathcal{H}}, \quad u \in \operatorname{Dom}(H),\ v \in \mathcal{R}(T).$$

5. Closed embeddings as absolute continuity of kernel operators

In this section we present an application of closed embedding to operator ranges, more precisely, we get an equivalent characterization of closed embedding of two given operator ranges in terms of the absolute continuity of the corresponding kernel operators. There is a recent interest in connection with the absolute continuity of nonnegative operators, e.g. [17] and the bibliography cited there. We first recall the definition and the basic properties of absolute continuity of two nonnegative bounded operators.

5.1. Parallel sum

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. If $A, B \in \mathcal{B}(\mathcal{H})$ are selfadjoint we write $A \leqslant B$ if $\langle Ah, h\rangle \leqslant \langle Bh, h\rangle$ for all $h \in \mathcal{H}$, the natural order relation (reflexive, antisymmetric, and transitive). We also denote by $\mathcal{B}(\mathcal{H})^+$ the convex strict cone of nonnegative operators.

Given $A, B \in \mathcal{B}(\mathcal{H})^+$ the *parallel sum* of A and B (originally defined by W.N. Anderson Jr. and R.J. Duffin [2] for matrices and then extended to bounded operators by P.A. Fillmore and J.P. Williams [15]) is

$$A : B = A^{1/2}C^*DB^{1/2}, \tag{5.1}$$

where C and D are the minimal bounded operators that produce the factorizations $A^{1/2} = (A + B)^{1/2}C$ and $B^{1/2} = (A + B)^{1/2}D$ (in fact, C and D can be characterized as certain pseudo-inverses). The following formula holds

$$A : B = \text{SO-}\lim_{\epsilon \searrow \infty} \big((A + \epsilon I)^{-1} + (B + \epsilon I)^{-1}\big)^{-1}, \tag{5.2}$$

where SO means that the limit should be taken with respect to the strong operator topology. T. Ando [4], and independently E.L. Pekarev and Yu.L. Shmulyan [25], got also the following formula

$$\big\langle (A : B)h, h\big\rangle = \inf\big\{\langle Ag, g\rangle + \big\langle B(h - g), h - g\big\rangle \bigm| g \in \mathcal{H}\big\}, \quad h \in \mathcal{H}. \tag{5.3}$$

Note that the binary operation of parallel addition is symmetric, more precisely, $A : B = B : A$, separately nondecreasing with respect to each argument, that is, if $A_1 \leqslant A_2$ then $A_1 : B \leqslant A_2 : B$, and that $0 \leqslant A : B \leqslant A, B$, but it is not (separately) additive.

5.2. Shorted operators and absolute continuity

Given $A, B \in \mathcal{B}(\mathcal{H})^+$, the *shorted operator* $[A]B \in \mathcal{B}(\mathcal{H})^+$ is by definition (cf. T. Ando [4], who generalized a previous definition introduced by W.N. Anderson Jr. and G.E. Trapp [3])

$$\big([A]B\big)h := \text{SO-}\lim_{n\to\infty} (nA) : B, \tag{5.4}$$

where the SO-limit exists because $(nA) : B \leqslant B$ and $(nA) : B \leqslant ((n + 1)A) : B$ for all $n \in \mathbb{N}$. Note that $[A]B \leqslant B$ but $[A]B$ may not be comparable with A.

Another formula to calculate $[A]B$ was obtained by H. Kosaki [20], namely, letting $P_{A,B}$ denote the orthogonal projection of $\mathcal{H}$ onto the closure of the subspace $\{h \in \mathcal{H} \mid B^{1/2}h \in \operatorname{Ran}(A^{1/2})\}$, where $\operatorname{Ran}(C)$ denotes the range of the operator C, then

$$[A]B = B^{1/2} P_{A,B} B^{1/2}. \tag{5.5}$$

For given $A, B \in \mathcal{B}(\mathcal{H})^+$ one says that A *uniformly dominates* B, in brief $B \leqslant_{\mathrm{u}} A$, if any of the following equivalent conditions holds (cf. [14]):

(i) There exists $t > 0$ such that $B \leqslant tA$, that is, $\langle Bh, h\rangle \leqslant t\langle Ah, h\rangle$ for all $h \in \mathcal{H}$.
(ii) $\operatorname{Ran}(B^{1/2}) \subseteq \operatorname{Ran}(A^{1/2})$.
(iii) There exists $X \in \mathcal{B}(\mathcal{H})$ such that $B^{1/2} = A^{1/2}X$.

This is a partial preorder relation (only reflexive and transitive) on $\mathcal{B}(\mathcal{H})^+$.

Theorem 5.1. *(See T. Ando [4].) Given $A, B \in \mathcal{B}(\mathcal{H})^+$, the following assertions are equivalent*:

(a) *There exists a sequence (B_n) in $\mathcal{B}(\mathcal{H})^+$ subject to the following conditions*:
  (aco1) *(B_n) is nondecreasing, in the sense that $B_1 \leqslant B_2 \leqslant \cdots \leqslant B_n \leqslant B_{n+1} \leqslant \cdots$;*
  (aco2) $\text{SO-}\lim_{n\to\infty} B_n = B$;
  (aco3) *For all $n \in \mathbb{N}$, $B_n \leqslant_{\mathrm{u}} A$ (i.e. for every $n \in \mathbb{N}$ there exists $t_n > 0$ such that $B_n \leqslant t_n A$).*
(b) *$B = [A]B$, that is, $\text{SO-}\lim_{n\to\infty}(nA) : B = B$.*
(c) *The linear manifold $\{h \in \mathcal{H} \mid B^{1/2}h \in \operatorname{Ran}(A^{1/2})\}$ is dense in $\mathcal{H}$.*

For given $A, B \in \mathcal{B}(\mathcal{H})^+$ one says that B is *A-absolutely continuous*, written $B \ll A$, if any of the equivalent assertions (a), (b), or (c) in Theorem 5.1 holds. The A-absolute continuity is additive in the sense that, if B and C are A-absolutely continuous then $B + C$ is A-absolutely continuous, but in general it is not hereditary, in particular, it is not a transitive relation. Clearly, if $B \leqslant_{\mathrm{u}} A$ then $B \ll A$. For the converse implication, the following result, that is implicit in [4], holds.

Proposition 5.2. *Let $A \in \mathcal{B}(\mathcal{H})^+$. The following assertions are equivalent*:

(i) *$\operatorname{Ran}(A)$ is closed.*
(ii) *For arbitrary $B \in \mathcal{B}(\mathcal{H})^+$, $B \leqslant_{\mathrm{u}} A$ if and only if $B \ll A$.*

$[A]B$ is also called the *Radon–Nikodym derivative* of B with respect to A, cf. T. Ando [4].

5.3. Closed embedding as absolute continuity

Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})^+$ associated to which there is the operator range $\mathcal{R}(A^{1/2})$, more precisely, $\mathcal{R}(A^{1/2}) = \operatorname{Ran}(A^{1/2})$ is a Hilbert space with the inner product $\langle A^{1/2}x, A^{1/2}y\rangle_A = \langle x, y\rangle_{\mathcal{H}}$. In addition, $\mathcal{R}(A^{1/2})$ is a Hilbert space continuously embedded in $\mathcal{H}$ and any Hilbert space $\mathcal{H}_+$ continuously embedded in $\mathcal{H}$ is of the form $\mathcal{R}(A^{1/2})$ for a unique kernel operator $A \in \mathcal{B}(\mathcal{H})^+$.

The main result of this section is that absolute continuity in the sense of T. Ando is just another facet of closed embedding. This result should be compared with Theorem 2.11. The equivalence of the assertions (1) and (2) in the next theorem is essentially a result of H. Kosaki, more precisely Lemma 3 in [20]; the proof of this equivalence follows the proof of Kosaki as well.

Theorem 5.3. *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. The following assertions are equivalent*:

(1) *$B \ll A$.*
(2) *$B^{1/2} = TA^{1/2}$ for some closed densely defined operator T in $\mathcal{H}$.*
(3) *$\operatorname{Ker}(A) \subseteq \operatorname{Ker}(B)$ and $\mathcal{R}(B^{1/2})$ is closely embedded in $\mathcal{R}(A^{1/2})$.*

Proof. Before starting the proof we claim that, without loss of generality we can assume that A is injective. This follows by showing that the condition $\operatorname{Ker}(A) \subseteq \operatorname{Ker}(B)$ holds in all assertions (1)–(3), and then by factoring out $\operatorname{Ker}(A)$ and restricting both operators to the reducing subspace $\mathcal{H} \ominus \operatorname{Ker}(A)$. Indeed, the condition $\operatorname{Ker}(A) \subseteq \operatorname{Ker}(B)$ is explicit in (3) and implicit in (2), so it remains only to show that it is a consequence of (1), as well. To this end, letting (B_n) be the sequence as in Theorem 5.1(a), note that $\operatorname{Ker}(A) \subseteq \operatorname{Ker}(B_n)$ for all $n \geqslant 1$ and hence

$$\operatorname{Ker}(A) \subseteq \bigcap_{n\geqslant 1} \operatorname{Ker}(B_n) \subseteq \operatorname{Ker}(B),$$

where the latter inclusion holds because B is the strong operator limit of (B_n). Thus, without loss of generality, throughout this proof we can assume that A, and hence $A^{1/2}$ as well, are injective.

(1) $\Rightarrow$ (2). Let (B_n) be the sequence as in Theorem 5.1(a). Since $\|B_n^{1/2}h\| \leqslant t_n^{1/2}\|A^{1/2}h\|$ for all $h \in \mathcal{H}$ and all $n \geqslant 1$ it follows that the map

$$\mathcal{H} \ni h \mapsto \left\|B^{1/2}h\right\| = \sup_{n\in\mathbb{N}}\left\|B_n^{1/2}h\right\| \in \mathbb{R}_+ \tag{5.6}$$

is lower continuous with respect to the norm $\|A^{1/2}\cdot\|$ on $\mathcal{H}$.

Let then $k_n = A^{1/2}h_n \to 0$ and $T_0k_n = B^{1/2}h_n \to y$ in $\mathcal{H}$, as $n \to \infty$. For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geqslant N$ we have

$$\|T_0k_m - T_0k_n\| = \left\|B^{1/2}(h_m - k_n)\right\| < \epsilon.$$

Letting $n \to \infty$ and taking into account of the lower-semicontinuity of the mapping in (5.6), for each fixed $m \geqslant N$ we get

$$\|T_0k_m\| = \left\|B^{1/2}h_m\right\| \leqslant \liminf_{n\to\infty}\left\|B^{1/2}(h_m - h_n)\right\| \leqslant \epsilon$$

whence, letting $m \to \infty$, we get $\|y\| \leqslant \epsilon$, and hence $y = 0$. This proves that T_0 is closable.

If T denotes the closure of T_0, we have $B^{1/2} = TA^{1/2}$.

(2) $\Rightarrow$ (1). Let T be a closed and densely defined operator in $\mathcal{H}$ such that $B^{1/2} = TA^{1/2}$, let $E_n = E[0, n]$, where E denotes the spectral measure of $|T| = (T^*T)^{1/2}$, and denote $B_n = A^{1/2}E_nT^*TE_nA^{1/2}$ for all $n \geqslant 1$. It is easy to see that the sequence (B_n) satisfies all the properties (aco1)–(aco3) in Theorem 5.1(a).

(2) $\Rightarrow$ (3). Let T be a closed densely defined operator in $\mathcal{H}$ such that $B^{1/2} = TA^{1/2}$. Then $A^{1/2}T^* \subseteq B^{1/2}$.

We consider the linear manifold

$$\mathcal{D} := \left\{A^{1/2}T^*x \;\middle|\; x \in \mathrm{Dom}\left(T^*\right)\right\} \tag{5.7}$$

and note that $\mathcal{D} \subseteq \mathcal{R}(A^{1/2}) \cap \mathcal{R}(B^{1/2})$. We first show that $\mathcal{D}$ is dense in $\mathcal{R}(B^{1/2})$. To see this, let $k \in \mathcal{R}(B^{1/2})$ be such that $k \perp \mathcal{D}$. Then, $k = B^{1/2}x$ for some $x \in \mathcal{H}$ and for any $y \in \mathrm{Dom}(T^*)$ we have

$$0 = \left\langle k, A^{1/2}T^*y\right\rangle_B = \left\langle B^{1/2}x, B^{1/2}y\right\rangle_B = \langle x, y\rangle$$

hence $x = 0$, since $\mathrm{Dom}(T^*)$ is dense in $\mathcal{H}$.

Now we consider the embedding j with domain $\mathcal{D}$ in $\mathcal{R}(B^{1/2})$ and range in $\mathcal{R}(A^{1/2})$, and show that it is closed. To see this, let (h_n) be a sequence of vectors in $\mathcal{D}$ such that it converges to h within the Hilbert space $\mathcal{R}(B^{1/2})$ and the sequence $jh_n = h_n$ converges to some $k \in \mathcal{R}(A^{1/2})$ within the Hilbert space $\mathcal{R}(A^{1/2})$. Then, for some sequence (x_n) in $\mathrm{Dom}(T^*)$ we have $h_n = A^{1/2}T^*x_n = B^{1/2}x_n$ for all $n \geqslant 1$, and $h = B^{1/2}x$ for some $x \in \mathcal{H}$. Since

$$\|h_n - h\|_B = \left\|B^{1/2}(x_n - x)\right\| = \|x_n - x\| \to 0 \quad \text{as } n \to \infty,$$

it follows that x_n converges to x within $\mathcal{H}$. On the other hand, since

$$\|jh_n - k\|_A = \|h_n - k\|_A = \left\|A^{1/2}T^*x_n - A^{1/2}y\right\| = \left\|T^*x_n - y\right\| \to 0 \quad \text{as } n \to \infty,$$

it follows that the sequence (T^*x_n) converges to y in $\mathcal{H}$. Thus, taking into account that T^* is closed, we get that $x \in \mathrm{Dom}(T^*)$ and $y = T^*x$, hence $h \in \mathcal{D}$ and $k = A^{1/2}T^*x = h = jh$, that is, j is closed.

(3) $\Rightarrow$ (2). Again, since $A^{1/2}$ is injective, the operator $T_0 = B^{1/2}A^{-1/2}$ is densely defined. We show that, assuming that $\mathcal{R}(B^{1/2})$ is closely embedded in $\mathcal{R}(A^{1/2})$, it follows that T_0^* is densely defined, hence T_0 is closable.

To this end, by assumption, there exists a linear manifold $\mathcal{D} \subseteq \mathcal{R}(A^{1/2}) \cap \mathcal{R}(B^{1/2})$ that is dense in $\mathcal{R}(B^{1/2})$ and the embedding $j : \mathcal{D} \to \mathcal{R}(A^{1/2})$ is closed, as an operator from $\mathcal{R}(B^{1/2})$ in $\mathcal{R}(A^{1/2})$. Observe now that

$$T_0^* = A^{-1/2}B^{1/2} \supseteq A^{-1/2}jB^{1/2},$$

where, in the rightmost factorization, the operator $B^{1/2} : \mathcal{H} \to \mathcal{R}(B^{1/2})$ is a coisometry and $A^{-1/2} : \mathcal{R}(A^{1/2}) \to \mathcal{H}$ is unitary, hence the operator $A^{-1/2}jB^{1/2}$ is densely defined. These show that T_0^* is densely defined, and hence that T_0 is closable. Finally, letting T the closure of T_0, it follows that $B^{1/2} = TA^{1/2}$. $\square$

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